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COMPRESSION OF A MULTILAYERED MEDIUM UNDER THE ACTION OF A VARIABLE EXTERNAL PRESSURE*

M.YU. IVANOV, V.K. KOROBOV, V.M. NIKOLAYENKO and K.P. STANYUKOVICH

The solutions of equations describing the system of waves that arise when a rigid body is compressed by means of a pressure that varies with time, are obtained in the acoustic approximation. The case when the compressed medium consists of two layers of different initial density is considered. The solutions obtained can be generalized to the case of the compression of a multilayer medium.

1. Let us consider the wave motions in a continuum under the action of a variable pressure $p = p(t)$ applied at the boundary of the medium. The equation of state for the rigid continuum is usually given in the form $p - p_0 = A(\rho^n - \rho_0^n) + B\rho T$, with the corresponding equation of the adiabatic curve $p - p_0 = A(\rho^n - \rho_0^n) + B_1(\rho^m \exp[(s - s_0)/c_v] - \rho_0^m)$. We shall replace the latter by the simpler equation of an adiabatic curve

$$p = A(s)\rho^\gamma - B \tag{1.1}$$

When the deformations of an elastic solid are small, Hooke's law $\sigma = k\varepsilon$ holds, where k is the bulk modulus ε is the deformation, $\varepsilon = (v - v_0)/v_0 = \rho_0/\rho - 1$, σ is the stress and $-\sigma = p$. We shall require that Eq.(1.1), in the linear approximation with respect to the deformation, shall be the same as Hooke's law $-\sigma = A\rho_0^\gamma - B - \gamma A\rho_0^\gamma\varepsilon$. This yields $A\rho_0^\gamma = B$, $\gamma A\rho_0^\gamma = k$. We note that the velocity of sound $c_0^2 = \gamma A\rho_0^{\gamma-1} = k/\rho$. Knowing the bulk modulus or the velocity of sound, and specifying the quantity γ, we can easily find the constants A and B for use in approximating (1.1).

We shall use the equations describing the propagation of the wave system in Lagrangian form, transforming them for convenience to the independent variables h, p:

$$u_p = r^N t_h, \quad x_h - v = x_p t_h / t_p$$
$$r_p = u t_p, \quad s_p = 0 \tag{1.2}$$
$$\left(h = \int_0^r \rho r^N\, dr, \quad x = r^{N+1}/(N+1)\right)$$

Here r is the Eulerian coordinate, h is the Lagrangian mass coordinate, u is the velocity and s is the entropy; the subscript denotes the derivative with respect to the corresponding variable; $N = 0, 1, 2$ for plane, cylindrical and spherical symmetry respectively.

Using the first and third equation of (1.2), we transform the second equation of (1.2) as follows:

$$x_h = v + u u_p \tag{1.3}$$

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Let us introduce the entalphy i. We know that $(\partial i/\partial p)_s = v$. By virtue of the last equation of (1.2) $(\partial i/\partial p)_s = (\partial i/\partial p)_h$, and therefore we can write (1.3) in the form

$$x_h = (i + u^2/2)_p \tag{1.4}$$

Eliminating the time from the first and third equation of (1.2) and transforming the result using (1.3), we obtain

$$\left(\frac{x_h - v}{ur^N}\right)_p = \left(\frac{r_p}{u}\right)_h \tag{1.5}$$

We shall solve the problem in the acoustic approximation, i.e. we shall assume that $\eta = u^2/(2i) \ll 1$. Let us estimate the accuracy of this approximation. For the approximation (1.1) $i = c^2/(\gamma - 1)$, and for the Riemann wave we have $u = 2(c - c_0)/(\gamma - 1)$. Then

$$\eta = \frac{2}{\gamma-1}\left[1 - \frac{c_0}{c}\right]^2 = \frac{2}{\gamma-1}\left[1 - \left(\frac{B}{p+B}\right)^\beta\right]^2 = \frac{2}{\gamma-1}\left[1 - \left(\frac{1}{\gamma p/k+1}\right)^\beta\right]^2, \quad \beta = \frac{\gamma-1}{2\gamma}$$

Suppose, for example, that $p = 10^{10}$ Pa, $k = 3\cdot 10^{11}$ Pa and $\gamma = 3$. Then $\eta \approx 10^{-3}$.
Moreover, we will assume that $p = p(t)$ everywhere except at the surfaces of discontinuity.

2. When $\eta \ll 1$, Eq.(1.4) takes the form $x_h = v$ and yields, after integration,

$$x = vh + \xi(p) \tag{2.1}$$

where $\xi(p)$ is an arbitrary function. Eq.(1.5) is now transformed (taking into account (2.1)), into $(r_p/u)_h = 0$ and yields, after integrating, $u/r_p = \eta(p)$ or

$$u = \frac{(v_p h + \xi_p)\,\eta(p)}{[(vh+\xi)(N+1)]^{N/(N+1)}} \tag{2.2}$$

where $\eta(p)$ is an arbitrary function. Next we need to find the functions $\xi(p)$ and $\eta(p)$. If the compressive pressure is caused by a moving solid, then the velocity of the surface of the compressed medium is equal to the velocity of motion of the solid u_s, and the pressure p_b at the boundary of the medium can easily be found since $p_b = f(u_s)$. If the compression is caused by the products of an explosion (PE), we have

$$p = p_e = p_b, \quad u = u_e = u_b \tag{2.3}$$

at the boundary between the compressed body and the PE.

The quantities with the subscript e refer in this section to the PE, and b denotes the quantities at the boundary. We assume that for the PE $p_e = A_e\rho_e^n - B_e$.

From the condition on the shock wave front appearing in the compressed solid we have, in the acoustic approximation /1/: $u_b = 2(c_0 - c_b)/(\gamma - 1)$. For the PE where we have a shock wave (when the explosion occurs on the right) or a rarefaction wave (when the explosion occurs on the left or is instantaneous), we have $u_{eb} = 2(c_{eb} - c_{e0})/(n - 1)$. (Here and henceforth the subscript 0 denotes the initial value). Expressing the velocity of sound in terms of the pressure, we obtain from (2.3) and the last two equations

$$\frac{n-1}{\gamma-1}\left[c_0 - \sqrt{\gamma A}\left(\frac{p_b+B}{A}\right)^{(\gamma-1)/(2\gamma)}\right] = c_{e0} - \sqrt{nA_e}\left(\frac{p_b+B_e}{A_e}\right)^{(n-1)/(2n)} \tag{2.4}$$

which yields $p_b(t)$. The initial velocity of sound in the PE $c_{e0}(t)$ is found from the interaction between the PE and the fixed wall /1/, from the following relations:

a) when the detonation travels towards the wall

$$p_e = p_0/(1+\tau)^3, \; p_0 = 2,4\rho_0 D^2/4, \; \tau = Dt/l$$

b) when the detonation travels away from the wall

$$p_e = \begin{cases} p_0, & 0 \leqslant \tau \leqslant 1, \\ p_0/\tau^3, & \tau > 1, \end{cases} \quad p_0 = \frac{2\rho_0 D^2}{27}, \quad \tau = \frac{Dt}{l}$$

c) when the detonation is instantaneous, as in case b), with

$$p_0 = \rho_0 D^2/8, \quad \tau = \sqrt{2}\,Dt/(2\sqrt{3}l)$$

For a standard explosive we have $\rho_0 = 1.6$ g/cm^3 and $D = 8000$ m/sec. In cases b) and c) the pressure change is described by two sets of conditions, therefore the first compression wave and the following waves will aslo be described by two sets of conditions. To make it simpler, it makes sense to approximate the pressure in all cases by the relation $p_e = P/(\tau_1 + \tau)^\alpha$ where $\tau = \beta Dt/l$; $P, \alpha, \beta, \tau_1 = \text{const}$, $\alpha \approx 3$. If a very rigid medium is compressed, then the displacement of its surface will be negligible and in this case, instead of solving the problem in the exact formulation, taking into account the rarefaction wave travelling along the PE, it will be more sensible to increase somewhat the value of the index α by putting $\alpha = n + \varepsilon$, $n \approx 3$, $\varepsilon \ll 1$.

Having obtained $p_b(t)$ from (2.4), we have

$$u_b = \frac{2}{\gamma - 1} c_0 \left[1 - \frac{\sqrt{\gamma A}}{c_0} \left(\frac{p_b + B}{A}\right)^{(\gamma-1)/(2\gamma)}\right] = \frac{\partial r_b}{\partial t}$$

Integrating this expression we obtain the law of motion of the surface of the compressed body $r_b = r_b(t)$. Carrying out the substitution $t = t(p)$ and using $p = p_b(t)$ we find, from (2.1),

$$\xi(p) = x_b(p) - v(p) h_b = r_b(p)^{N+1}/(N+1) - v(p) h_b$$

From (2.2) we obtain

$$\eta(p) = \frac{u_b(p)}{[(vh_b + \xi)(N+1)]^{N/(N+1)}} = \frac{2}{\gamma - 1} \cdot \frac{[c_0 - c(p)](v_p h_b + \xi_p)}{[(vh_b + \xi)(N+1)]^{N/(N+1)}}$$

Here $h_b = \rho_0 R^{N+1}/(N+1)$ where $2R$ is the size of the compressed body.

The velocity of the compression wavefront in the acoustic approximation is

$$D = \frac{u_b - c_b - c_0}{2} = -\frac{c_0}{2(\gamma - 1)} \left[(\gamma + 1)\frac{c_b}{c_0} - (3 - \gamma)\right] = \frac{dr_{1f}}{dt}$$

Integrating the above expression, we obtain the law of motion of the wavefront

$$r_{1f} = R - \left[\frac{\gamma + 1}{2(\gamma - 1)} \sqrt{\gamma A} \int_0^t \left(\frac{p_b + B}{A}\right)^{\frac{\gamma-1}{2\gamma}} dt - \frac{3 - \gamma}{2(\gamma - 1)} c_0 t\right] \tag{2.5}$$

The compression wave will reach, at some $t = t_0$, the centre of symmetry $r_{1f}(t_0) = 0$ and a reflected wave will appear. Since, when $h = 0$, we must have $r \equiv 0$ at the centre, it follows from (2.1) that behind the reflected wave $\xi \equiv 0$ or $x = v_2 h_2$. We shall denote the parameters in front and behind the reflected wave by the indices 1 and 2 respectively.

At the reflected wavefront we have, in the acoustic approximation, /1/ $u_2 + 2c_2/(\gamma - 1) = 2c_1/(\gamma - 1) - u_1$. Taking into account the fact that $u_1 = 2(c_0 - c_1)/(\gamma - 1)$, we obtain

$$u_2 = 2(2c_1 - c_0 - c_2)/(\gamma - 1) \tag{2.6}$$

The velocity of the reflected wavefront is

$$D_2 = \frac{u_2 + c_2 + u_1 + c_1}{2} = \frac{\gamma + 1}{2(\gamma - 1)} c_1 - \frac{3 - \gamma}{2(\gamma - 1)} c_2 = \frac{dr_{2f}}{dt} \tag{2.7}$$

Since we also have

$$u_2 = (v_1 - v_2) D_2/v_1 = (1 - \rho_1/\rho_2) D_2 \tag{2.8}$$

we obtain from (2.6)-(2.8)

$$\frac{u_2}{1 - (c_1/c_2)^{2/(\gamma-1)}} = \frac{2(2c_1 - c_0 - c_2)}{(\gamma - 1)[1 - (c_1/c_2)^{2/(\gamma-1)}]} = \frac{(\gamma + 1)}{2(\gamma - 1)} c_1 - \frac{3 - \gamma}{2(\gamma - 1)} c_2$$

The above equations yield $c_2 = c_2(t)$ (and $p_2 = p_2(t)$); $u_2 = u_2(t)$. After this, integrating (2.7) we find the law of motion of the reflected wavefront

$$r_{2f}(t) = \int_{t_0}^t \left[\frac{\gamma + 1}{2(\gamma - 1)} c_1 - \frac{3 - \gamma}{2(\gamma - 1)} c_2\right] dt$$

The condition that the solutions match at the front yields $r^{N+1}/(N+1) = v_1 h_1 + \xi = v_2 h_2$, and this gives the Lagrangian coordinates before and behind the front $h_1(t)$ and $h_2(t)$, which will obviously be different. Further, making the substitution $t = t(p)$ and using the relation $p = p_2(t)$ obtained, we find

$$\eta(p) = \frac{r_{2p}}{u_2} = \frac{(\gamma - 1) v_{2p} h_2(p)}{2(2c_1 - c_0 - c_2)[v_2 h_2(p)(N+1)]^{N/(N+1)}}$$

which solves the problem of the reflected wave completely. Putting $u_2 = 0$, we find the pressure and density at the centre of symmetry. Having solved the equation $r_{2f}(t) = r_b(t)$, we find the time t_R at which the reflected wave emerges at the surface of the compressed body. After this emergence another reflected wave appears, moving towards the centre of symmetry of the body. Fig.1 shows this system of waves.

Let us discuss a possible refinement. Since $r_p = u\eta(p)$, it follows from the third relation of (1.2) that $t_p = \eta(p)$ or $t = \int \eta(p)\,dp + \theta(h)$, and not simply $t = t(p)$, as was written before in an approximation manner. It is theoretically easy, although relatively complicated to carry out in practice, to assume that $t = \pi(p) + \theta(h)$ and solve the problems of the compression and reflected waves anew, but we shall not carry out these calculations here.

3. Let us consider a more complicated problem, namely the compression of a two-layer body. Let a symmetrical body radius R_1 contain inside it another body of radius R_2, with density $\rho_2 \neq \rho_1$, in close contact with the first body. Let the equation of state of the medium

of which the inner body is made, be $p_2 = A_2\rho_2^\delta - B_2$.

When a compression wave travelling from the surface of the first medium arrives at the second medium at the instant t_1, the second medium starts to compress. We find the time t_1 from (2.5), by putting $r_{1f}(t) = R_2$. Two waves appear at this instant of time. One wave travels to the right in the first medium, and the second wave travels to the left in the second medium. We shall denote the parameters of the media using letters with two indices. The first number will denote the medium, and the second number the wave.

Let us write the equations of state of both media in the form

$$p_1 = A_1\left(\frac{c_1^2}{\gamma A_1}\right)^{\gamma/(\gamma-1)} - B_1, \quad p_2 = A_2\left(\frac{c_2^2}{\delta A_2}\right)^{\delta/(\delta-1)} - B_2 \tag{3.1}$$

The following conditions must hold at the surface separating the two media:

$$p_1 = p_2 = p^*, \quad u_1 = u_2 = u^* \tag{3.2}$$

and we also have

$$u_1 = u_{11} + 2(c_{12} - c_{11})/(\gamma - 1) = 2(c_{12} + c_{10} - 2c_{11})/(\gamma - 1) \tag{3.3}$$
$$u_2 = 2(c_{20} - c_{21})/(\delta - 1)$$

Eliminating c_{12} and c_{21} from (3.1) and (3.3) and using conditions (3.2), we obtain the following relation for determining the velocity of the boundary surface u^*:

$$A_1\left[\frac{(\gamma-1)u^*/2 + 2c_{11} - c_{10}}{\sqrt{\gamma A_1}}\right]^{2\gamma/(\gamma-1)} - B_1 = A_2\left[\frac{c_{20} - (\delta-1)u^*/2}{\sqrt{\delta A_2}}\right]^{2\delta/(\delta-1)} - B_2$$

The law of motion of the boundary surface is obtained by integrating

$$r^*(t) = \int_{t_1}^{t} u^*\,dt$$

After this we find $c_{12}(t)$, $c_{21}(t)$ and $p_{12}(t)$, $p_{21}(t)$ from (3.3). The Lagrangian coordinates at the boundary surface are

$$h_1^* = \rho_{10}x^* = \rho_{10}R_2^{N+1}/(N+1), \quad h_2^* = 2\rho_{20}x^* = \rho_{20}R_2^{N+1}/(N+1)$$

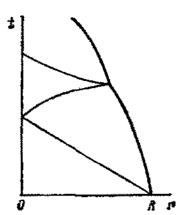

Fig.1

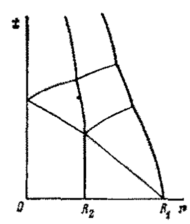

Fig.2

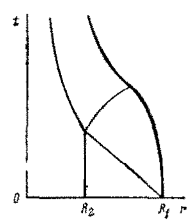

Fig.3

Now using solution (2.1) for both media $r^{*N+1}/(N+1) = x^* = v_{12}h_1^* + \xi_{12} = v_{21}h_2^* + \xi_{21}$ and carrying out, as before, appropriate substitutions $t = t(p)$, we find

$$\xi_{12}(p_1) = x^*(p_1) - v_{12}h_1^* = x^*(p_1) - v_{12}\rho_{10}R_2^{N+1}/(N+1)$$
$$\xi_{21}(p_2) = x^*(p_2) - v_{21}h_2^* = x^*(p_2) - v_{21}\rho_{20}R_2^{N+1}/(N+1)$$

Finally we obtain

$$\eta_{12}(p_1) = u^*(p_1)/r_p^*(p_1), \quad \eta_{21}(p_2) = u^*(p_2)/r_p^*(p_2)$$

The wave 1 will travel to the centre and will be reflected, the reflected wave (wave 3) will travel towards the surface catching up with wave 2. From then on, various situations may arise. Depending on the magnitude of the ratio R_2/R_1, wave 3 reflected from the centre will catch up with wave 2 before it emerges at the surface, or wave 2 will reach the surface earlier and in this case wave 3 will interact with the unloading wave travelling from the outer surface. Fig.2 depicts this system of waves.

In conclusion, we shall consider the case when the second medium is replaced by a cavity. In this case we must put $\rho_2 = 0$, $p_2 = 0$ in all relations written for the second medium. When the first compression wave reaches the cavity, a rarefaction wave will travel along the first medium towards its outer surface, and the treatment of this wave is elementary. The dispersed material of the first medium will spread inside the cavity, and cleavage may occur at the boundary, at certain corresponding values of the pressure.

If the cavity is filled with a porous material, the situation remains practically unchanged for the first medium, but the porous medium will be compressed. The process can be easily studied if the parameters of the first wave are known. Fig.3 shows the wave pattern for such cases.

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FORCED OSCILLATIONS IN IMPERFECT AND STATICALLY LOADED SHELLS*

A.YU. POPOV

The influence of small, non-axisymmetric imperfections in the middle surface and of a static load, on the amplitude of the forced oscillations of shells of revolution of zero curvature, is studied. For this puropse, shells acted upon by a mixed load, namely a static and dynamic load, are computed. The problem of a mixed load applied to an ideal shell is reduced to the problem of statics for a shell containing imperfections which vary with time. The amplitude-frequency relations are constructed for the flexure of statically loaded shells within the range of the lowest resonance frequencies. It is shown that in the case of statically loaded shells these relations differ essentially from those for load-free shells. The greatest increase in the amplitude of forced oscillations is observed in forms where the number of waves in parallel corresponds to the lowest frequencies.

In investigating the influence of static loads or form imperfections on the dynamic behaviour of shells, the greatest attention has been given, as a rule, to the change in the resonance frequencies. In practice, it is essential to know the behaviour of the oscillation amplitude under static loads, or resulting from form imperfections, and this is important when studying the dynamic behaviour of loaded shells as a whole.

One of the methods of solving the problem of the statics or dynamics of imperfect shells is based on the introduction of irregularity parameters into the initial system of equations. A linear system of equations is chosen as the initial system. The small-parameter method is then used, just as was done in problems of the statics of imperfect shells /1-3/. The same approach can be used in the problem of shells under a mixed load, and such a problem has been studied experimentally**.(**Solodilov V.E. Study of the natural oscillations of shells using holographic interferometry. Candidate Dissertation, Moscow, Inst. problem mechaniki, Akad. Nauk SSSR, 1980).

Let us consider the forced oscillations of a shell of revolution with an imperfect middle surface, excited by an axisymmetric harmonic load. We shall describe the imperfections in the middle surface of the shell using functions of the type $w_0 = \varepsilon r(z) \cos m\varphi$ where w_0 is the initial sag, z is the meridianal coordinate, φ is the circular coordinate, m is the number of waves in parallel, and ε is a number, small compared with the relative thickness of the shell. We shall write the coefficients of the solution of the system of equations describing the forced oscillations of an arbitrary shall, in the form of series in powers of the small parameter ε. After substituting the coefficients and the solution into the initial system, the latter splits into several subsystems. The zeroth approximation corresponds to the problem of the forced oscillations of a perfect shell of revolution. Every subsequent approximation is constructed by integrating the system of equations for the perfect shell of revolution, with various right-hand sides in the equations of equilibrium as well as in the geometrical relations. Thus the analysis of a shell with small, non-axisymmetric imperfections,

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